JEE Main 2023 (Memory based)

25 January 2023 - Shift 2

Answer & Solutions

MATHEMATICS

1. The value of $\sum_{k=0}^{6} {}^{51-k}C_3$ is

A. ${}^{52}C_4 - {}^{46}C_4$
B. ${}^{52}C_4 - {}^{45}C_4$
C. ${}^{51}C_4 - {}^{45}C_4$
D. ${}^{51}C_4 - {}^{46}C_4$

Answer (B)
Solution:

$$\sum_{k=0}^{6} {}^{51-k}C_3 = {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + \cdots + {}^{45}C_3 + {}^{45}C_4 - {}^{45}C_4$$

As we know that ${}^{45}C_3 + {}^{45}C_4 = {}^{46}C_4$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 + {}^{46}C_3 + {}^{46}C_4 - {}^{45}C_4$

As we know that ${}^{46}C_3 + {}^{46}C_4 = {}^{47}C_4$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 + {}^{47}C_4 - {}^{45}C_4$

Continuing the same process, we have

$= {}^{52}C_4 - {}^{45}C_4$

2. If $f(x) = 2x^n + \lambda$ and $f(4) = 133,\ f(5) = 255$, then sum of positive integral divisors of $f(3) - f(2)$ is :

A. 60
B. 22
C. 40
D. 6

Answer (A)

Solution:

As $f(4) = 133$ and $f(x) = 2x^n + \lambda$

$\Rightarrow 2 \cdot 4^n + \lambda = 133 \cdots (i)$

As $f(5) = 255$

$\Rightarrow 2 \cdot 5^n + \lambda = 255 \cdots (ii)$

Subtracting Equation (i) from Equation (ii)

$2 \cdot (5^n - 4^n) = 122$

$\Rightarrow n = 3$

$f(x) = 2x^3 + \lambda$

$f(3) - f(2) = 2(3^3 - 2^3) = 38$

Divisors are: 1,2,19,38

Sum$= 1 + 2 + 19 + 38 = 60$

3. If $\left|\frac{z+2i}{z-i}\right| = 2$ is a circle, then centre of circle is:

A. $(0,\ 0)$
B. $(0,\ 2)$
C. $(2,\ 0)$
D. $(-2,\ 0)$

Answer (B)

Solution:

$(z+2i)(\overline{z}-2i) = 4(z-i)(\overline{z}+i)$

$\Rightarrow\ z\overline{z} + 2i\overline{z} - 2iz + 4 = 4z\overline{z} - 4i\overline{z} + 4iz + 4$

$\Rightarrow\ 3z\overline{z} - 6i\overline{z} + 6iz = 0$

$\Rightarrow\ z\overline{z} - 2i\overline{z} + 2iz = 0$

$\therefore$ Centre $\equiv 2i$

$i.e. (0, 2)$

4. If $\frac{dy}{dt} + \alpha y = \gamma \cdot e^{-\beta t}$, then $\lim_{t\to\infty} y(t)$, where $\alpha > 0,\ \beta > 0,\ \gamma > 0,\ \alpha \neq \beta$, is equal to:

A. 0
B. 1
C. Does not exist
D. αβ

Answer (A)

Solution:

$\frac{dy}{dt} + \alpha y = \gamma e^{-\beta t}$
Integrating factor (I. F.) $= e^{\alpha t}$
Solution of L.D.E.
$ye^{\alpha t} = \gamma \int e^{(\alpha-\beta)t} dt$
$\Rightarrow ye^{\alpha t} = \frac{\gamma}{\alpha-\beta} \cdot e^{(\alpha-\beta)t} + C$
$\Rightarrow y(t) = \frac{\gamma}{\alpha-\beta} \cdot e^{-\beta t} + C \cdot e^{-\alpha t}$
$\Rightarrow \lim_{t\to\infty} y(t) = 0$

5. If $(p \to q)\nabla(p\Delta q)$ is tautology, then operator ∇, Δ denotes:

A. $\Delta \to OR$ and $\nabla \to AND$
B. $\Delta \to AND$ and $\nabla \to OR$
C. $\Delta \to AND$ and $\nabla \to AND$
D. $\Delta \to OR$ and $\nabla \to OR$

Answer (D)

Solution:
$(p \to q)\nabla(p\Delta q) \equiv T$
Only if ∇ is OR and Δ is OR

6. The number of numbers between 5000 & 10000 formed by using the digits $1,3,5,7,9$ without repetition is equal to:

A. 120
B. 72
C. 12
D. 6

Answer (B)

Solution:

The leftmost digit can be chosen in 3 ways i.e. $5,7,9$

Now, the digits can be chosen from remaining digits for remaining places in 4 ways, 3 ways, 2 ways and 1 way.

Total numbers$= 3 \times 4 \times 3 \times 2 \times 1 = 72$

7. If $f(x) = \log_{\sqrt{m}}\left(\sqrt{2}(\sin x - \cos x) + m - 2\right)$, the range of $f(x)$ is $[0,\ 2]$, then the value of m is:

A. 3
B. 4
C. 5
D. None

Answer (C)

Solution:

We know that $\sin x - \cos x \in \left[-\sqrt{2},\ \sqrt{2}\right]$

$\log_{\sqrt{m}}\left((\sin x - \cos x) + m - 2\right) \in \left[\log_{\sqrt{m}}(m-4), \log_{\sqrt{m}} m\right]$

$\log_{\sqrt{m}}(m-4) = 0$ & $\log_{\sqrt{m}} m = 2$

$\Rightarrow m = 5$

8. If A be a symmetric matrix and B & C are skew symmetric matrices of same order, then:

A. $A^{13} \cdot B^{26} - B^{26} \cdot A^{13}$ is symmetric.
B. $AC - A$ is symmetric.
C. $A^{13} \cdot B^{26} - B^{26} \cdot A^{13}$ is symmetric.
D. $AC - A$ is skew symmetric.

Answer (C)

Solution:

A is symmetric $\Rightarrow A^{13}$ is symmetric.

B is skew-symmetric $\Rightarrow B^{26}$ is skew-symmetric.

Now, let $A^{13} = P$ and $B^{26} = Q$

$A^{13} \cdot B^{26} - B^{26} \cdot A^{13}$

$= PQ - QP$

Now, $(PQ - QP)^T = (PQ)^T - (QP)^T = Q^T \cdot P^T - P^T \cdot Q^T$

$\Rightarrow QP - PQ = -(PQ - QP)$

$\Rightarrow (A^{13}B^{26} - B^{26}A^{13})^T = -(A^{13}B^{26} - B^{26}A^{13})$

$\therefore (A^{13}B^{26} - B^{26}A^{13})$ is skew-symmetric matrix.

9. Consider the function $f(x) = \begin{cases} (1+|\cos x|)^{\frac{\lambda}{|\cos x|}} , & x < \frac{\pi}{2} \\ \mu \quad , & x = \frac{\pi}{2} \\ e^{\frac{\cot 6x}{\cot 4x}} \quad , & x > \frac{\pi}{2} \end{cases}$

A. $\lambda = \frac{2}{3}, \mu = e^{\frac{2}{3}}$

B. $\lambda = e^{\frac{2}{3}}, \mu = \frac{2}{3}$

C. $\lambda = \frac{3}{2}, \mu = e^{\frac{3}{2}}$

D. $\lambda = e^{\frac{3}{2}}, \mu = \frac{3}{2}$

Answer (A)

Solution:

$\lim_{x \to \frac{\pi}{2}^-} f(x) = e^{\lim_{x \to \frac{\pi}{2}^-} |\cos x| \cdot \frac{\lambda}{|\cos x|}} = e^{\lambda}$

$\Rightarrow \mu = e^{\lambda}$

$\lim_{x \to \frac{\pi}{2}^+} f(x) = e^{\lim_{x \to \frac{\pi}{2}^+} \frac{\cot 6x}{\cot 4x}} = e^{\lim_{h \to 0} \frac{\cot 6h}{\cot 4h}} = e^{\frac{2}{3}}$

$\Rightarrow \mu = e^{\frac{2}{3}}, \lambda = \frac{2}{3}$

10. Two dice are rolled. If the probability the sum of the numbers on dice is n, where $n-2, \sqrt{3n}, n+2$ are in geometric progression, is $\frac{x}{48}$, then the value of x is:

A. 4

B. 12

C. 7

D. 3

Answer (A)

Solution:

As given, $\left(\sqrt{3n}\right)^2 = (n-2)(n+2)$

$\Rightarrow 3n = n^2 - 4$

$\Rightarrow n = 4, \quad n = -1$ (Not possible)

Favourable outcomes $(1,2), (2,1), (2,2)$

Total outcomes $6 \times 6 = 36$

Given $\frac{3}{36} = \frac{x}{48}$

$\Rightarrow x = 4$

11. Let $\vec{a} = -\hat{\imath} - \hat{\jmath} + \hat{k}$ such that $\vec{a} \cdot \vec{b} = 1$ and $\vec{a} \times \vec{b} = \hat{\imath} - \hat{\jmath}$ then $\vec{a} - 6\vec{b}$ equals:

A. $3\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)$

B. $\hat{\imath} + \hat{\jmath} + \hat{k}$

C. $2(\hat{i}+\hat{j}+\hat{k})$
D. $4(\hat{i}+\hat{j}+\hat{k})$

Answer (A)

Solution:

$\vec{a}\times\vec{b}=\hat{i}-\hat{j}$
$\Rightarrow \vec{a}\times(\vec{a}\times\vec{b})=(-\hat{i}-\hat{j}+\hat{k})\times(\hat{i}-\hat{j})$
$\Rightarrow (\vec{a}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{a})\vec{b}=\hat{i}+\hat{j}+2\hat{k}$
$\Rightarrow 3\vec{b}=-2\hat{i}-2\hat{j}-\hat{k}$
$\therefore \vec{a}-6\vec{b}=-\hat{i}-\hat{j}+\hat{k}-(-4\hat{i}-4\hat{j}-2\hat{k})$
$\Rightarrow \vec{a}-6\vec{b}=3\hat{i}+3\hat{j}+3\hat{k}=3(\hat{i}+\hat{j}+\hat{k})$

12. $16\int_1^2 \frac{dx}{x^3(x^2+2)^2}$ is equal to:

A. $\frac{11}{12}+\ln 4$
B. $\frac{11}{12}-\ln 4$
C. $\frac{11}{6}-\ln 4$
D. $\frac{11}{6}+\ln 4$

Answer (C)

Solution:

$I=\int \frac{dx}{x^3(x^2+2)^2}$
$=\frac{1}{4}\int\frac{x}{x^2+2}dx+\frac{1}{4}\int\frac{x}{(x^2+2)^2}dx-\frac{1}{4}\int\frac{1}{x}dx+\frac{1}{4}\int\frac{1}{x^3}dx$
$I=\frac{\ln(x^2+2)}{8}-\frac{1}{8(x^2+2)}-\frac{\ln x}{4}-\frac{1}{8x^2}$
$16\int_1^2\frac{dx}{x^3(x^2+2)^2}=2\ln 6-2\ln 3-4\ln 2+\frac{11}{6}$
$=\frac{11}{6}-\ln 4$

13. If $A=\begin{bmatrix}\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{bmatrix}$ and $B=\begin{bmatrix}1 & -i\\ 0 & 1\end{bmatrix}$. If $M=A^TBA$, then the matrix $AM^{2023}A^T$ is:

A. $\begin{bmatrix}1 & -2023i\\ 0 & 1\end{bmatrix}$
B. $\begin{bmatrix}1 & 2023i\\ 0 & 1\end{bmatrix}$
C. $\begin{bmatrix}1 & -2023i\\ 0 & -1\end{bmatrix}$
D. $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Answer (A)

Solution:

$AA^T=I$
$M=A^TBA$
$AM^{2023}A^T=A\underbrace{(A^TBA)(A^TBA)(A^TBA)\cdots(A^TBA)}_{2023\ times}A^T$

$$= B^{2023}$$

$$AM^{2023}A^T = B^{2023}$$

$$B^2 = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2i \\ 0 & 1 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 1 & -2i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3i \\ 0 & 1 \end{bmatrix}$$

$$B^{2023} = \begin{bmatrix} 1 & -2023i \\ 0 & 1 \end{bmatrix}$$

14. The remainder when $(2023)^{2023}$ is divided by 35 is __________.

Answer (7)

Solution:

$2023 \equiv -7(35)$

$(2023)^2 \equiv 14(35)$

$(2023)^4 \equiv -14(35)$

$(2023)^{16} \equiv -14(35)$

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$(2023)^{2020} \equiv -14(35)$

and $(2023)^3 \equiv 7(35)$

$\therefore (2023)^{2023} \equiv 7(35)$

$\therefore$ remainder $= 7$

15. If $\int_{\frac{1}{3}}^{3} |\ln x| dx = \frac{m}{n} \ln\left(\frac{n^2}{e}\right)$, then value of $m^2 + n^2 - 5$ is equal to _______.

Answer (20)

Solution:

$$\int_{\frac{1}{3}}^{3} |\ln x| dx = \int_{\frac{1}{3}}^{1} -\ln x \, dx + \int_{1}^{3} \ln x \, dx$$

$$= -[(x \ln x - x)]_{\frac{1}{3}}^{1} + [(x \ln x - x)]_{1}^{3}$$

$$= \frac{2}{3} + \frac{1}{3}\ln\frac{1}{3} + 3\ln 3 - 2$$

$$= \frac{4}{3}(\ln 9 - \ln e)$$

$$= \frac{4}{3}\ln\left(\frac{3^2}{e}\right)$$

$\therefore m = 4, n = 3$

$m^2 + n^2 - 5 = 16 + 9 - 5 = 20$

16. A triangle is formed with x –axis, y –axis & line $3x + 4y = 60$. A point $P(a, b)$ lies strictly inside the triangle such that a is a positive integer and b is a multiple of $'a'$. The numbers of such points (a, b) is ______.

Answer (31)

Solution:

x	y	Points	No. of points
1	$\frac{57}{4}$	$(1,1),(1,2),\ldots,(1,14)$	14
2	$\frac{27}{2}$	$(2,2),(2,4),\ldots,(2,12)$	6
3	$\frac{51}{4}$	$(3,3),\ (3,6),\ldots,(3,12)$	4
4	12	$(4,4),(4,8)$	2
5	$\frac{45}{4}$	$(5,5),\ (5,10)$	2
6	$\frac{21}{2}$	$(6,6)$	2
7	$\frac{39}{4}$	$(7,7)$	1
8	9	$(8,8)$	1
9	$\frac{33}{4}$	0	0
10	$\frac{15}{2}$	0	0

Total points $= 14+6+4+2+2+1+1+1$

$= 31$

17. If $a,\ b, \frac{1}{18}$ are in G.P and $\frac{1}{10}, \frac{1}{a}, \frac{1}{b}$ are in A.P, then the value of $a+180b$ is __________.

Answer (20)

Solution:

$b^2 = \frac{a}{18}, \frac{2}{a} = \frac{1}{10} + \frac{1}{b}$

$\Rightarrow a = \frac{20b}{10+b}$ or $18b^2 = \frac{20}{10+b}$

$b = 0$ or $9b = \frac{10}{10+b}$

$90b + 9b^2 = 10$

$\Rightarrow 9b^2 + 90b - 10 = 0$

$\therefore a + 180b = 18b^2 + 180b = 20$

18. In a city, 25% of the population is smoker, and a smoker has 27 times more chance of being diagnosed with lung cancer. A person is selected at random and found to be diagnosed with lung cancer. If the probability of him being smoker is, $\frac{k}{40}$. Then the value of k is __________.

Answer (36)

Solution:

Probability of a person being a smoker $= \frac{1}{4}$

Probability of a person being nonsmoker $= \frac{3}{4}$

$P\left(\frac{S}{SC}\right) = \frac{\frac{1}{4} \cdot 27P}{\frac{1}{4} \cdot 27P + \frac{3}{4}P} = \frac{27}{30} = \frac{9}{10} = \frac{36}{40}$

$\Rightarrow k = 36$